IMPORTANT CALENDAR CHANGES

Close *Tuesday*: 15.3, 15.4

Exam 2 is THURSDAY!!!

10.3,13.4,14.1,14.3,14.4,14.7,15.1-4

(now includes center of mass)

15.1-15.3 Summary

We have 3 ways to describe a region:

"Top/Bottom":

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

"Left/Right":

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

"Inside/Outside":

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

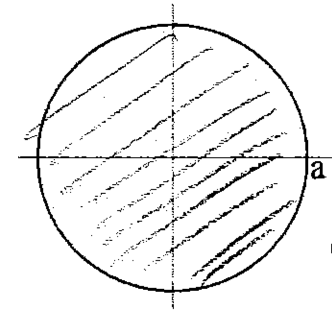
Entry Task:

Let D be the circular disc of radius *r* = *a*.

Evaluate the integrals.

$$\iint_D 1\,dA = ?$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\,dA = ?$$



$$\boxed{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le a \end{matrix}}$$

$$\iint_D 1\,dA = \int_0^{2\pi}\int_0^a 1 \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^a \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a^2 \, d\theta = \frac{1}{2} a^2 \theta \Big|_0^{2\pi}$$

$$= \frac{1}{2} a^2 (2\pi) = \boxed{\pi a^2 = \text{AREA OF THE CIRCLE}}$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\, dA$$

$$\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

$u = a^2 - r^2$

$du = -2r\,dr$

$-\frac{1}{2r}\,du = dr$

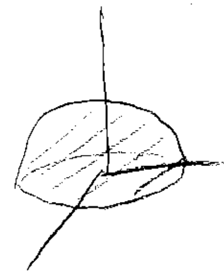
$$\int_0^{2\pi}\int_{a^2}^{0} \sqrt{u}\;\cancel{r}\;\frac{1}{-2\cancel{r}}\,du\;d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_{a^2}^{0} \right) d\theta$$

$$= \int_0^{2\pi} \left[0 - -\frac{1}{3}(a^2)^{3/2} \right] d\theta$$

$$= \frac{1}{3} a^3 \theta \Big|_0^{2\pi}$$

$$= \boxed{\frac{2}{3}\pi a^3}$$ = HALF THE VOLUME OF A SPHERE



$z = \sqrt{a^2 - x^2 - y^2}$ ← UPPER HALF-SPHERE

$\Rightarrow\; z^2 = a^2 - x^2 - y^2$

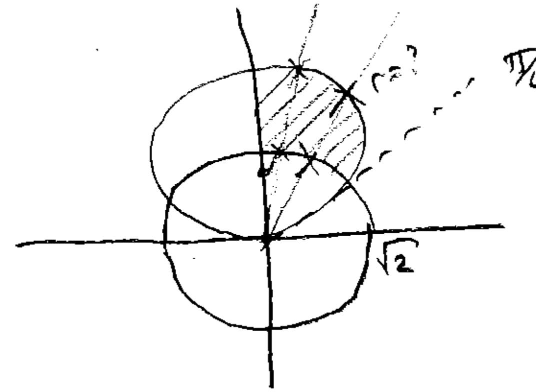
$\Rightarrow\; x^2 + y^2 + z^2 = a^2$ SPHERE

Old Exam Question

Find the area of the region in the first quadrant that is outside $x^2 + y^2 = 2$ and inside the circle $x^2 + y^2 = 2y$.

$$\boxed{\iint_D 1\, dA = ???}$$

$$x^2+y^2-2y = 0$$
$$x^2+y^2-2y+1 = 1$$
$$x^2+(y-1)^2 = 1$$



POLAR!

$$x^2+y^2=2 \Rightarrow r^2=2 \Rightarrow r=\sqrt{2}$$
$$x^2+y^2=2y \Rightarrow r^2=2r\sin\theta \Rightarrow r=2\sin\theta$$

INTERSECT

$$2\sin\theta = \sqrt{2}$$
$$\sin\theta = \sqrt{2}/2$$
$$\theta = \pi/4$$

$$\pi/4 \le \theta \le \pi/2$$
$$\sqrt{2} \le r \le 2\sin\theta$$

$$\iint_D 1\, dA = \int_{\pi/4}^{\pi/2}\int_{\sqrt{2}}^{2\sin\theta} 1\cdot r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2}r^2\Big|_{\sqrt{2}}^{2\sin\theta} d\theta$$

HALF-ANGLE!!!

$$= \int_{\pi/4}^{\pi/2} \frac{1}{2}4\sin^2\theta - \frac{1}{2}(\sqrt{2})^2\, d\theta = \int_{\pi/4}^{\pi/2} 2\sin^2\theta - 1\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} 2\left(\frac{1}{2}(1-\cos(2\theta)\right) - 1\, d\theta = \int_{\pi/4}^{\pi/2} \cancel{1} - \cos(2\theta) \cancel{-1}\, d\theta = -\frac{1}{2}\sin(2\theta)\Big|_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{2}\sin(\pi) - -\frac{1}{2}\sin(\pi/2) = \boxed{1/2}$$

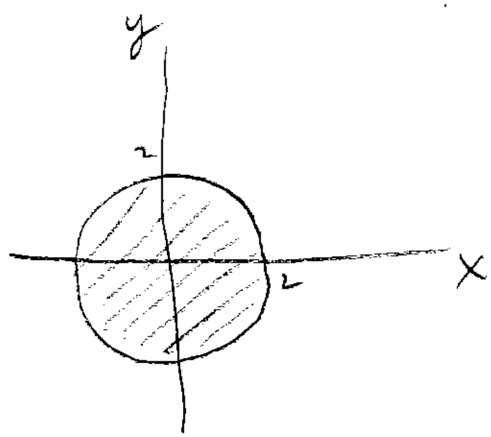
Old Exam Question

Find the volume of the solid that is inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 1$, and below the surface $z + y = 3 + x^2 + y^2$.

$$z = 3 + x^2 + y^2 - y$$

$$\iint_D 3 + x^2 + y^2 - y \, dA - \iint_D 1 \, dA$$

$$= \iint_D 2 + x^2 + y^2 - y \, dA$$



$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\int_0^{2\pi} \int_0^2 (2 + r^2 - r\sin\theta) \, r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^2 2r + r^3 - r^2 \sin\theta \, dr \, d\theta$$

$$\int_0^{2\pi} r^2 + \frac{1}{4}r^4 - \frac{1}{3}r^3 \sin\theta \Big|_0^2 \, d\theta$$

$$\int_0^{2\pi} 4 + 4 - \frac{8}{3}\sin\theta \, d\theta$$

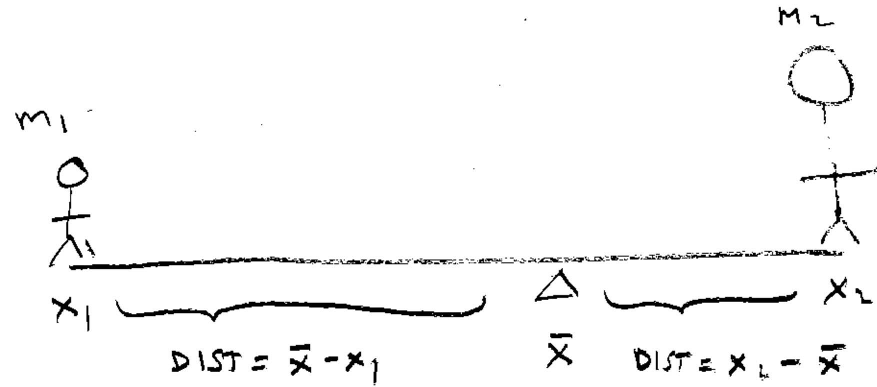
$$8\theta + \frac{8}{3}\cos\theta \Big|_0^{2\pi}$$

$$\left(8(2\pi) + \frac{8}{3}(1)\right) - \left(0 + \frac{8}{3}(1)\right)$$

$$= \boxed{16\pi}$$

15.4 Center of Mass

Motivation "the see-saw"



New App: Consider a thin plate (*lamina*) with density at each point given by

$$\rho(x, y) = \text{mass/area} \quad (\text{kg/m}^2).$$

We will see that the center of mass (centroid) is given by

$$\bar{x} = \frac{\text{"Moment about y"}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

$$\bar{y} = \frac{\text{"Moment about x"}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

LAW OF LEVEL $\Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$

$$\Rightarrow m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$\Rightarrow m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\Rightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

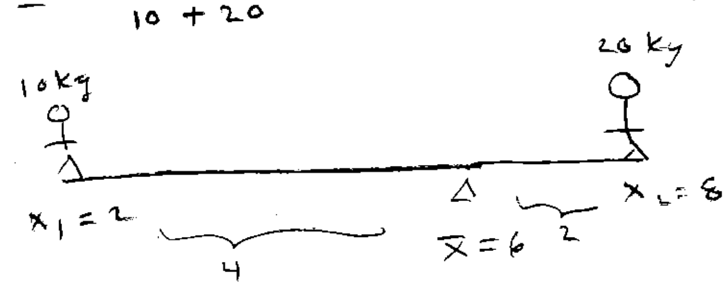
"MOMENT ABOUT y-AXIS"

Ex) $m_1 = 10$, $m_2 = 20$

$x_1 = 2$, $x_2 = 8$

$$\bar{x} = \frac{(10)(2) + (20)(8)}{10 + 20} = \frac{20 + 160}{30} = \frac{180}{30} = 6$$

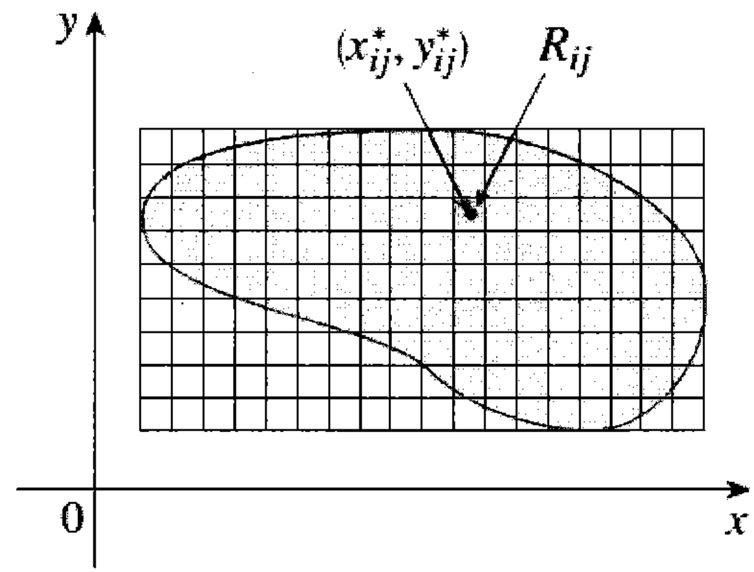
"TOTAL MASS"



In general: If you are given ***n* points**
$(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with
corresponding masses $m_1, m_2, \ldots, m_n$
then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R \; x\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R \; y\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

Example:

Consider a 1 by 1 m square metal plate. The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.

Find the center of mass.

Side note:

The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as distance from the y-axis.

In other words, the plate gets heavier at a constant rate from left-to-right.

TOTAL MASS $= \int_0^1 \int_0^1 kx\,dx\,dy = \int_0^1 \frac{k}{2}x^2\Big|_0^1 dy$

$= \frac{k}{2}\int_0^1 1\,dy = \frac{k}{2}\,y\Big|_0^1 = \boxed{k/2 = M}$

MOMENT ABOUT y $= \int_0^1 \int_0^1 x\,(kx)\,dx\,dy = \int_0^1 \frac{k}{3}x^3\Big|_0^1 dy$

$= \frac{k}{3}\int_0^1 1\,dy = \boxed{\frac{k}{3} = M_y}$

MOMENT ABOUT x $= \int_0^1 \int_0^1 y\,(kx)\,dx\,dy = \int_0^1 \frac{k}{2}\,y\,x^2\Big|_0^1 dy$

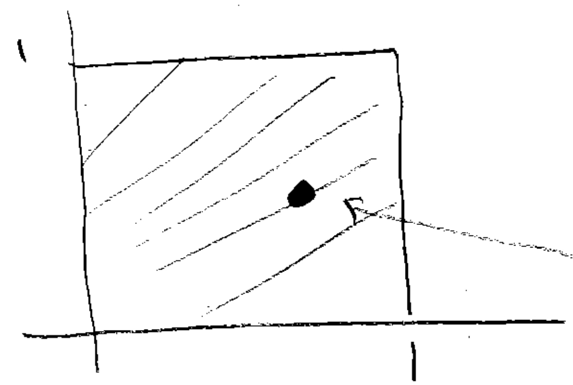
$= \frac{k}{2}\int_0^1 y\,dy = \frac{k}{2}\,\frac{1}{2}y^2\Big|_0^1 = \boxed{\frac{k}{4} = M_x}$

$\bar{x} = \frac{M_y}{M} = \frac{k/3}{k/2} = 2/3$

$\bar{y} = \frac{M_x}{M} = \frac{k/4}{k/2} = 1/2$

CENTER OF MASS

$\boxed{\left(\frac{2}{3}, \frac{1}{2}\right)}$



Example) If $k = 15$

then $p(\frac{1}{3}, 1) = 15 \cdot \frac{1}{3} = 5$ $\frac{kg}{m^2}$

$p(\frac{2}{3}, 1) = 15 \cdot \frac{2}{3} = 10$ $\frac{kg}{m^2}$

$p(1, 1) = 15 \cdot 1 = 15$ $\frac{kg}{m^2}$

SAME ANSWER NO MATTER WHAT K IS.

Translations:

Density proportional to the dist. from…

…the y-axis -- $p(x,y) = kx.$

…the x-axis -- $p(x,y) = ky.$

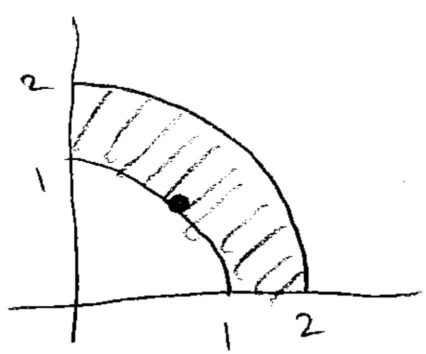
…the origin -- $p(x,y) = k\sqrt{x^2+y^2}.$

Density proportional to the <u>square</u> of the distance from the origin:

$$p(x,y) = k(x^2+y^2).$$

Density <u>inversely</u> proportional to the distance from the origin:

$$p(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$



Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin. Find the center of mass.

$$p(x,y) = k\sqrt{x^2+y^2}$$

$$M = \iint_D p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 kr\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \frac{k}{3} r^3\Big|_1^2\,d\theta$$
$$= \int_0^{\pi/2} \frac{k}{3}(8-1)\,d\theta$$
$$= \frac{7}{3}k\,\theta\Big|_0^{\pi/2} = \frac{7\pi}{6}k$$

$$M_y = \iint_D x\,p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 r\cos\theta\, kr\cdot r\,dr\,d\theta$$
$$= k\int_0^{\pi/2}\cos\theta\ \frac{1}{4}r^4\Big|_1^2\,d\theta = \frac{k}{4}(16-1)\ \sin\theta\Big|_0^{\pi/2}$$
$$= \frac{15k}{4}$$

$$M_x = \iint_D y\,p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 r\sin\theta\, kr\,r\,dr\,d\theta$$
$$= k\int_0^{\pi/2}\sin\theta\ \frac{1}{4}r^4\Big|_1^2\,d\theta = \frac{k}{4}(16-1)\left(-\cos\theta\Big|_0^{\pi/2}\right) = 15k/4$$

$$\bar{x} = \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \approx 1.02313 \qquad \bar{y} = \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \approx 1.02313$$